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by

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By Lipman Bers

1. Introduction

Let S be a Riemann surface and f a homeomorphism of S onto another Riemann surface S'. If f is, in terms of local parameters, of class C_1 and has a positive Jacobian, then the deviation of this mapping from conformality can be measured, at each point p of S, by the ratio $K_p > 1$ of the axes of an infinitesimal ellipse into which f takes an infinitesimal circle located at p. We set $K[f] = \sup K_p$. The "overall dilitation" K[f] can be defined also for a somewhat wider class of "quasiconformal" homeomorphisms.

Teichmüller's theorem asserts that given any orientation preserving homeomorphism $f : S \rightarrow S'$ between two closed Riemann surfaces of genus $g > 1$, there exists, among all mappings homotopic to f, a unique "extremal" f_o which minimizes K[f]. Furthermore, this extremal homeomorphism can be described analytically by two uniquely determined holomorphic quadratic differentials, $\phi(z)dz^2$ defined on S and $\psi(z')dz'^2$ defined on S'. In terms of local parameters $\zeta = \xi + i\eta$ and $\zeta' = \xi' + i\eta'$ such that $d\zeta^2 = \phi dz^2$, $d\zeta'^2 = \psi dz'^2$ the mapping f_o can be written in the form : $\xi' = k\xi$, $\eta' = \eta$. Thus the extremal mapping is real analytic, except at the isolated zeros of the quadratic differentials, and the "local dilitation" K_p is a constant. Finally Teichmüller calls

$\log K[f_0]$ the distance between the Riemann surfaces S and S', relative to the homotopy class of f, and asserts that the space of "topologically determined" or, as we shall say, "marked" Riemann surfaces, with this distance function, is homeomorphic to the Euclidean space of 6g - 6 dimensions.

In his first memoir [22] of 1939, Teichmüller succeeded only in proving the uniqueness part of this theorem. Later [23] he gave an existence proof based on the continuity mrthod. The idea of this proof is quite simple, but the details are rather cumbersome. Another existence proof was found several years ago by Ahlfors [1].

In view of the importance and the beauty of Teichmüller's result[1] it seems worthwhile to give here a new version of the existence proof. Our arrangement of the argument preserves the logical structure of Teichmüller's proof; the details are carried out differently. More precisely, we work with the most general definition of quasiconformality, we rely on the theory of partial differential equations in some crucial parts of the argument, and we make use of a simple set of moduli for marked Riemann surfaces. In order to make this presentation self-contained, we shall also sketch Teichmüller's uniqueness proof.

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1 Cf. Plutarch [19], "...it does not of necessity follow that, if the work delights you with its grace, the one who wrought it is worthy of your esteem."

Actually the Teichmüller theory refers to a more general situation. One considers not only closed Riemann surfaces, but also surfaces bounded by a finite number of analytic curves, and the homeomorphisms considered may be required to take on prescribed values at a finite number of prescribed points. But it is known (see Ahlfors' paper for details) that this more general situation can be reduced to the case of closed surfaces by means of doubling and by introducing appropriate two- or four- sheeted covering surfaces. Thus, for the sake of brevity, we shall be concerned only with closed surfaces.

2. Generalized Derivatives

It is convenient to base the definition of quasiconformality on the concept of generalized derivatives due to Sobolev [21] and Friedrichs [7]. We recall the definition of these derivatives.

A) Let $f(x,y)$, $g(x,y)$, $h(x,y)$ be measurable functions[2] defined in a domain D of the z plane ($z=x+iy$) which belong to L_2 over every compact subset of D. One says that g and h are the generalized partial derivatives of f with respect to x and y, respectively, and one writes $g = f_x$, $h = f_y$ if the following three conditions are satisfied.

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2 Functions differing on a set of (two dimensional) measure zero will be considered identical.

(i) For every C_1 function ω which vanishes outside a compact set in D

$$\iint_D (f\omega_x + g\omega)dxdy = \iint_D (f\omega_y + h\omega)dxdy = 0 .$$

(ii) For every compact subset $\Sigma \subset D$ there exists a sequence of C_1 functions $f^{(j)}$ such that

$$\iint_\Sigma (|f - f^{(j)}|^2 + |g - f_x^{(j)}|^2 + |h - f_y^{(j)}|^2)dxdy \rightarrow 0 .$$

(iii) The function $f(x,y)$ is absolutely continuous in one variable for almost all (relevant) values of the other, and the ordinary partial derivatives of f are almost everywhere equal to g and h, respectively.

The basic theorem on generalized derivatives asserts that each of the conditions (i) - (iii) implies the other two. The implications (ii) $\rightarrow$ (iii) and (iii) $\rightarrow$ (i) are almost obvious. To prove the implication (i) $\rightarrow$ (ii) one uses the so-called mollifiers or some other smoothing device. Details will be found, for instance, in Smirnov's course [20].

B) The rules for operating with generalized derivatives follow easily from the definition and need not be spelled out. We mention explicitely only a lemma on convergence.

<u>Assume that $\{f^{(j)}\}$ is a sequence of functions defined in a domain D and having there generalized derivatives. Assume also that the Dirichlet integrals of the functions $f^{(j)}$ are uniformly bounded and that the sequence $\{f^{(j)}\}$ converges in</u>

the L_2 sense. Then the limit function $f = \lim f^{(j)}$ has generalized derivatives which are weak limits, in the L_2 sense, of the corresponding generalized derivatives of a properly chosen subsequence $\left\{f^{(j_n)}\right\}$.

Indeed, local weak compactness of Hilbert space implies the existence of a subsequence $\left\{f^{(j_n)}\right\}$ such that the sequences $\left\{f_x^{(j_n)}\right\}$ and $\left\{f_y^{(j_n)}\right\}$ have weak limits g and h, and definition (i) of generalized derivatives shows that $g = f_x$, $h = f_y$.

C) The formal derivatives of a function $w(z)$ with respect to the complex variables $z = x = i y$ and $\bar{z} = x - i y$ are defined by the formulas

$$2w_z = w_x - i\, w_y, \quad 2w_{\bar{z}} = w_x + i\, w_y .$$

These derivatives obey the usual rules of calculus and they may be understood either in the classical or in the generalized sense.

Let $w = u + iv$ be a complex-valued function defined in a domain D. The equation $w_{\bar{z}} = 0$ is equivalent to the Cauchy-Riemann system for the functions u,v. It is known and easy to verify that if the function w has generalized derivatives and $w_{\bar{z}} = 0$ almost everywhere, then w is an analytic function of z.

3. Quasiconformal Mappings of Plane Domains

In this section we define the concept of quasiconformality for univalent functions in the plane. This definition will be extended to mappings of Riemann surfaces in §7.

A) Let $w = w(z)$ be a homeomorphism of a plane domain D, of class C_1 and with a positive Jacobian. Consider an infinitesimal circle located at some point z of D; the mapping w takes it into an infinitesimal ellipse. The ratio $K(z)$ of the major axis of this ellipse to the minor axis measures the deviation of the mapping from conformality; we have

$$K = \frac{\max\limits_{0 \le \theta \le 2\pi} | \cos \theta \cdot w_x + \sin \theta \cdot w_y |}{\min\limits_{0 \le \theta \le 2\pi} | \cos \theta \cdot w_x + \sin \theta \cdot w_y |}$$

and, setting

$$K = \frac{1 + k}{1 - k} ,$$

also

$$k = \frac{|w_{\bar{z}}|}{|w_z|} .$$

The mapping is called quasiconformal if $K(z)$ is uniformly bounded, that is if there is a number $k < 1$ such that

$$(3.1) \qquad | w_{\bar{z}} | \le k \, | w_z | \qquad (k < 1)$$

in D. Setting $w = u + i v$, this inequality may also be

written as

$$(3.1') \qquad u_x^2 + u_y^2 + v_x^2 + v_y^2 \leq (K + \frac{1}{K})\ (u_x v_y - u_y v_x) .$$

This definition of quasiconformality is, however, too restrictive. Thus Teichmüller permits the mapping to fail to be continuously differentiable at isolated points, and the most interesting quasiconformal mappings actually possess such singular points. A natural definition of quasiconformality, occurring for the first time in Morrey [16], is obtained by interpreting the derivatives in (3.1) as generalized derivatives. Thus we shall call a homeomorphism w quasiconformal (more precisely, k quasiconformal) if w has generalized derivatives satisfying the inequality (3.1) almost everywhere.

B) Another definition of quasiconformality, free of all explicit differentiability requirements, is due to Ahlfors [1] and Pfluger [18]. A topological rectangle R with modulus m is, by definition, a conformal image of the Euclidean rectangle $o \leq \xi \leq m$, $o \leq \eta \leq 1$. We write $m = \text{mod } R$. By virtue of Riemann's mapping theorem a topological image of a topological rectangle is again a topological rectangle. According to the "geometric" definition a homeomorphism w of a plane domain D is k quasiconformal if

$$(3.2) \qquad \frac{\text{mod } w\ (R)}{\text{mod } R} \leq \frac{1 + k}{1 - k}$$

for every topological rectangle $R \subset D$.

A classical result of Grotzsch [8,9] shows that this inequality holds for a C_1 mapping which is quasiconformal in the sense of A). It is also known that (3,2) holds for quasiconformal mappings with generalized derivatives. This can be seen, for instance, by repeating Grotzsch's argument and using the result of §4, C). On the other hand, in view of a theorem by Mori [14], inequality (3.2) has as its consequences the existence of generalized derivatives and the differential inequality (3.1), as I have shown elsewhere [2]. The two definitions are thus equivalent.

C) A direct consequence of the geometric definition is the fact that the composite of a k_1 quasiconformal mapping with a k_2 quasiconformal one is k quasiconformal, where

$$\frac{1+k}{1-k} = \frac{1+k_1}{1-k_1} \cdot \frac{1+k_2}{1-k_2} .$$

In particular, a k quasiconformal mapping remains so if followed or preceded by a conformal transformation.

Another consequence of the geometric definition is the fact that the inverse mapping of a k quasiconformal mapping is also k quasiconformal.

D) One should mention another definition of quasiconformality which goes back to Lavrent'ev [11] (cf. Volkoviskii [24] and the references given there). Let w be a homeomorphism of D and set, for $z \in D$,

$$\hat{K}(z) = \limsup_{r \to 0} \frac{\max\limits_{0 \le \theta \le 2\pi} |w(z+re^{i\theta}) - w(z)|}{\min\limits_{0 \le \theta \le 2\pi} |w(z+re^{i\theta}) - w(z)|}$$

According to the "differential-geometrical" definition w is called quasiconformal if $\hat{K}$ is uniformly bounded; it is called k quasiconformal if

$$\hat{K}(z) \le \frac{1+k}{1-k}$$

almost everywhere. The equivalence of this definition with the previous two follows from the work of Pesin [17] (who considers a more general case) and Jenkins [10].

E) We state now the basic inequality for quasiconformal mappings. Let $w(z)$ be a k quasiconformal homeomorphism of the unit disc onto itself which leaves the origin fixed. There exist positive constants A, α depending only on k such that

$$(3.3) \qquad |w(z_1)-w(z_2)| \le A|z_1-z_2|^{\alpha}, \quad |z_1-z_2| \le A|w(z_1)-w(z_2)|^{\alpha}.$$

Thus every quasiconformal homeomorphism of the open unit disc onto itself is automatically a homeomorphism of the closed unit disc.

This result occurs first in Morrey [16]. The sharp exponent is $\alpha = (1-k)/(1+k)$ (Lavrent'ev [12], Ahlfors). The best value of the constant A, independent of k, is 16 (Mori [15]).

Inequality (3.3) implies, of course, similar inequalities

for k quasiconformal mappings normalized in a different way, for instance, by prescribing the images of three points on the unit circle.

4. Beltrami Equations

Our proof of Teichmüller's existence theorem will be based on the theory of Beltrami equations with bounded measurable coefficients. We collect here the necessary results, which are due to Morrey (cf. also Bers and Nirenberg [3], Boyarskii [5]). Selfcontained proofs will be found in a forthcoming paper by Bers and Nirenberg [4]. We remark that it would be easy, though somewhat awkward, to reword our proof in such a way that it would involve only Beltrami equations with coefficients which are real analytic except at isolated points. For such equations the results stated below are classical.

A) By a Beltrami equation for a complex-valued function $w(z) = u + iv$ we mean an equation of the form

$$w_{\bar{z}} = \mu(z) w_z \tag{4.1}$$

where $\mu(z)$ is a measurable function satisfying the inequality

$$|\mu(z)| \leq k < 1 \,. \tag{4.2}$$

By a solution we mean a continuous function having generalized derivatives which satisfy (4.1) almost everywhere.

We associate with μ the Riemannian metric

$$ds^2 = E\,dx^2 + 2\,F\,dx\,dy + G\,dy^2 = \lambda(z)|dz + \mu\,d\bar{z}|^2, \quad \lambda > 0$$

(and note that every Riemannian metric can be written in this form). Equation (4.1) expresses the fact that w is a mapping conformal with respect to this metric. One sees this by writing (4.1) in real form; the equations read

$$Wu_x = -Fv_x + Ev_y, Wu_y = -Gv_x + Fv_y$$

where $W^2 = EG - F^2$.

We state now the main theorems on Beltrami equations.

B) _If_ $w_1(z)$ _is a solution of the Beltrami equation_ (4.1) _and_ $f(w)$ _an analytic function, then_ $w_2(z) = f(w_1(z))$ _is also a solution of_ (4.1). _Conversely, if_ $w_1(z)$ _and_ $w_2(z)$ _are two solutions of the same Beltrami equation, defined in the same domain, and_ $w_1(z)$ _is a homeomorphism, then_ $w_2(z) = f(w_1(z))$ _where_ $f(w)$ _is an analytic function._

We remark that if $\mu(z)$ satisfies a Hölder condition, then all solutions of (4.1) are of class C_1, and for such classical solutions the assertion is, of course, trivial.

C) _A homeomorphic solution_ $w(z) = u + iv$ _of a Beltrami equation takes (two-dimensionally) measurable sets into measurable sets, and for every such set_ $\triangle$

$$\iint_{\triangle} (u_x v_y - u_y v_x)\, dxdy = \iint_{w(\triangle)} dudv \,.$$

D) <u>Let $\mu(z)$ be a measurable function defined in the unit disc and satisfying (4.2). There exist solutions $w = u + iv$ of the Beltrami equation (4.1) which map the unit disc topologically onto itself.</u>

For every such solution we have, by (3.1') and C)

$$(4.3) \quad \iint_{|z|<1} (u_x^2+u_y^2+v_x^2+v_y^2)\, dxdy \le 2\frac{1+k^2}{1-k^2} \iint_{|z|<1} (u_x v_y - u_y v_x)\, dxdy = 2w\, \frac{1+k^2}{1-k^2} \,.$$

It follows from B) that all solutions considered may be expressed in terms of a single solution w_o by the formula

$$w(z) = e^{i\theta}\, \frac{w_o(z) - \zeta}{1 - \bar{\zeta}\, w_o(z)}$$

Where θ is real and $|\zeta| < 1$.

E) The connection between Beltrami equations and quasiconformal mappings is contained in the following self-evident but important statement: Every homeomorphic solution of a Beltrami equation satisfying (4.2) is a k quasiconformal mapping; conversely, every k quasiconformal mapping is a solution of some Beltrami equation satisfying (4.2).

It follows from §3, E) that every solution considered in D) realizes a homeomorphism of the closed unit disc onto itself. Thus a solution may be determined uniquely by

prescribing the images of three boundary points, or the images of one interior point and one boundary point.

F) We conclude by establishing the continuous dependence of the solutions discussed in D) on the coefficient μ.

Let there be given a sequence of measurable functions $\mu_j(z)$ defined in the unit disc, and assume that

(4.4) $|\mu_j| \leq k < 1, j = 1,2,\dots \; ; \; \mu_j(z) \to \mu(z)$ a.e.

Let $w^{(j)}(z)$ denote the uniquely determined solution of the Beltrami equation $w_{\bar{z}}^{(j)} = \mu_j w_z^{(j)}$ which maps the unit disc topologically onto itself and satisfies the conditions

(4.5) $$w^{(j)}(0) = 0, w^{(j)}(1) = 1 .$$

Then the sequence $\{w^j\}$ converges uniformly to the uniquely determined solution w of the Beltrami equation (4.1) which maps the unit disc onto itself leaving the points 0,1 fixed.

Proof. The function $w^{(j)}$ are all k quasiconformal and it follows from §3, E) and Arzela's theorem that a properly chosen subsequence $\{w^{(j_n)}\}$ converges uniformly to a function w which is also a homeomorphism of the closed unit disc onto itself leaving the points 0,1 fixed. By D) the Dirichlet integrals of the functions $w^{(j)}$ are uniformly bounded. Hence, by §2, B) the function w has generalized derivatives and we may assume, selecting if need be a subsequence, that $w_x^{(j_n)} \to w_x, w_y^{(j_n)} \to w_y$ weakly, in the L_2 sense. Then

also $w_z^{(j_n)} \longrightarrow w_z, w_{\bar z}^{(j_n)} \longrightarrow w_{\bar z}$ weakly, and by (4.4), $\mu_j w_z^{(j_n)} \longrightarrow \mu w_z$ weakly. Hence w is a solution of (4.1). Thus w is uniquely determined (by E), so that the selection of a subsequence $w^{(j_n)}$ was unnecessary.

Remark. It is clear from the proof that the normalization condition (4.5) could be replaced by any other condition which normalizes a conformal mapping of the unit disc onto itself.

5. Representation of Riemann surfaces by Fuchsian groups.

Before extending the definition of quasiconformality to mappings of Riemann surfaces we recall the representation of Riemann surfaces by fixed-point-free discontinuous groups of non-Euclidean motions. For the sake of brevity, such groups will be called simply Fuchsian groups.

A) We denote by U the upper half-plane of the complex z plane. In U we introduce the Poincaré metric based on the line element $ds = |dz| y^{-1}$; this makes U into a model of the non-Euclidean plane. The geodesics of this metric (non-Euclidean straight lines) are arcs of Euclidean circles or straight lines orthogonal to the real axis. The non-Euclidean distance between two points will be denoted by $\langle z_1, z_2 \rangle$.

We denote by W the group of all homeomorphisms of U

onto itself, by W_o the subgroup of orientation preserving homeomorpgisms, and by W^* the subgroup of W consisting of mappings which are continuous and one-to-one on the extended real axis. The mapping

$$Q(z) = \frac{z - i}{z + i} \tag{5.1}$$

takes U into the unit disc. An element $T \varepsilon W$ belongs to W^* if and only if QTQ^{-1} is a homeomorphism of the closed unit disc.

The subgroup of W_o which leaves the non-Euclidean distance invariant is the group $H \subset W^*$ of non-Euclidean motions. The elements of H are linear transformations

$$T(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \; (\alpha, \beta, \gamma, \delta \text{ real}, \; \alpha\delta - \beta\gamma > 0) .$$

By 1 we denote the identity transformation.

We topologize W by requiring $T_n \rightarrow T$ to mean that $T_n(z) \rightarrow T(z)$ uniformly on compact subsets of U . We topologize W^* by requiring $T_n \rightarrow T$ to mean that $QT_nQ^{-1} \rightarrow QTQ^{-1}$ uniformly on the closed unit disc. For elements of H both convergence concepts coincide.

B) An element $T \varepsilon H$ distinct from 1 is called a non-Euclidean translation if it leaves a non-Euclidean line (axis of the translation) invariant. An equivalent condition is that T leaves exactly two points on the real axis fixed; these points are the end-points of the axis of T . Of the two

fixed points one, which we denote by $\sigma = \sigma(T)$, is repelling, the other, denoted by $\tau = \tau(T)$, is attracting: $T^{-n}(z) \to \sigma$ and $T^n(z) \to \tau$ for $n = 1,2,\ldots$. A translation may be written, uniquely, in the form

$$\frac{T(z) - \sigma}{T(z) - \tau} = \lambda \frac{z - \sigma}{z - \tau} , \quad \lambda > 1 ,$$

with an obvious interpretation of this formula in case either σ or τ is the point ∞. The number $\lambda = \lambda(T)$ is called the invariant of T.

C) By a Fuchsian group we shall mean here a subgroup $G \subset H$ such that 1 is an isolated point of G and $1 \neq T \in H$ implies that $T(z) \neq z$ for all $z \in U$. Let G be such a group. For $z \in U$ the equivalence class $[z]_G$ of z under G consists of all points $T(z), T \in G$; it is known to be a discrete point set. Let U/G denote the set of equivalence classes; the natural mapping $\Pi_G : U \to U/G$ sends z into $[z]_G$. We make U/G into a Riemann surface by requiring Π_G to be, locally, a conformal homeomorphism.

Uniformization theory implies that every Riemann surface which is not the sphere, the sphere punctured at one or two points, or a closed surface of genus 1, can be represented as U/G . The representation is not unique, but U/G_1 and U/G_2 are conformally equivalent Riemann surfaces if and only if there is an $A \in H$ such that $G_1 = AG_2A^{-1}$.

D) A fundamental region of a Fuchsian group G is a set $\Sigma \subset U$ such that $\Pi_C(\Sigma) = U/G$ and the restriction of Π_G to the interior of Σ is one-to-one. The Riemann surface U/G is closed (compact) if and only if G has a compact fundamental region.

A Fuchsian group G with a compact fundamental region Σ consists of 1 and of non-Euclidean translations.

For the sake of completeness, we prove this well known result. Let $1 \neq T \in G$ and set $d = \inf < z, T(z) >$, $z \in U$. For every $z \in U$ there is a $\zeta \in \Sigma$ and an $A \in G$ such that $z = A(\zeta)$, and $<A(\zeta), TA(\zeta)> = <\zeta, A^{-1}TA(\zeta)>$. Hence $d = \inf <\zeta, A^{-1}TA(\zeta)>$, $\zeta \in \Sigma$, $A \in G$. Since Σ is compact and G discrete, there are elements A_o and $T_o = A_o^{-1}TA_o$ in G and a point ζ_o such that $d = <\zeta_o, T_o(\zeta_o)>$. Set $\zeta_1 = T_o(\zeta_o), \zeta_2 = T_o^2(\zeta_o)$ and let z_o be the midpoint of the ono-Euclidean segment (ζ_o, ζ_1). Then $z_1 = T_o(z_o)$ is the midpoint of the non-Euclidean segment (ζ_1, ζ_2) and $<z_o, z_1> \leq <z_o, \zeta_1> + <\zeta_1, z_1> = d$. On the other hand $<z_o, z_1> \geq d$. Thus $<z_o, z_1> = d$ and the three distinct points $\zeta_o, \zeta_1, \zeta_2$ lie on a non-Euclidean line ℓ which is invariant under T_o. Hence T_o is a translation and so is $T = A_oT_oA_o^{-1}$.

E) Let G be a Fuchsian group and $S = U/G$. By $\pi_1(S,p)$ we denote the fundamental group of S at a point $p \in S$. The elements of $\pi_1(S,p)$ are the homotopy classes $[\alpha]_p$ of

closed curves α beginning and ending at p, and $[\alpha]_p[\beta]_p = [\alpha\beta]_p$. An isomorphism $\omega : \pi_1(S,p) \to \pi_1(S,q)$ will be called allowable if there is a curve γ on S leading from q to p such that $\omega([\alpha]_p) = [\gamma\alpha\gamma^{-1}]_q$. In particular, every inner automorphism of $\pi_1(S,p)$ is allowable.

Let $z_o \in U$, $p_o = \Pi_G(z_o)$. For $T \in G$ let α_T denote a curve in U leading from z_o to $T(z_o)$ and set $\alpha'_T = \Pi_G(\alpha_T)$, $\Lambda_{G,z_o}(T) = [\alpha'_T]_{p_o}$. Then Λ_{G,z_o} is a canonical isomorphism of G onto $\pi_1(S,p_o)$.

Lemma. <u>Let</u> z_1 <u>and</u> z_2 <u>be two points of</u> U, $p_1 = \Pi_G(z_1)$, $p_2 = \Pi_G(z_2)$, <u>and let</u> $\lambda : G \to G$ <u>and</u> $\omega : \pi_1(S,p_1) \to \pi_1(S,p_2)$ <u>be isomorphisms onto connected by the relation</u>

$$\lambda = \Lambda^{-1}_{G,z_2}\,\omega\,\Lambda_{G,z_1} .$$

<u>Then</u> λ <u>is an inner automorphism if and only if</u> ω <u>is allowable.</u>

The proof may be omitted.

6. <u>Continuous mappings of Riemann surfaces</u>

In this section we recall the representation of continuous mappings of Riemann surfaces by mappings of the upper half-plane. This is needed in order to apply the results on Beltrami equations.

A) Let S and S' be two Riemann surfaces. Two homomorphisms $\lambda:\pi_1(S,p) \to \pi_1(S',p')$ and $\nu:\pi_1(S,q) \to \pi_1(S',q')$ will be called equivalent if there exist allowable isomorphisms (cf. §5, E)) ω and ω' such that $\nu = \omega'\lambda\,\omega$.

Let $f : S \to S'$ be a continuous mapping. For every $p \in S$ it induces a homomorphism $f_p:\pi_1(S,p) \to \pi_1(S',f(p))$ defined by $f_p([\alpha]_p) = [f(\alpha)]_{f(p)}$. For any two points p and q of S the homomorphisms f_p and f_q are equivalent.

Two continuous mappings of S into S' are homotopic if and only if they induce equivalent homomorphisms of the fundamental groups. We need not assume this known theorem since a proof, for the case of surfaces with a hyperbolic universal covering surface, will result from the following considerations.

B) Let G and G' be Fuchsian groups, $w:U \to U$ and $f:U/G \to U/G'$ continuous mappings and $\chi:G \to G'$ a homomorphism. We say that w induces f if

$$(6.1) \qquad f\prod_G = \prod_{G'} w .$$

We say that w induces χ if

$$(6.2) \qquad wT = \chi(T)w \quad \text{for} \quad T \in G .$$

Note that, by (6.2), the element $\chi(T) \in G'$ is uniquely determined by T and w, since an element of G' is determined by what it does to a single point of U. Also, if (6.2) holds for some mapping $\chi:G — G'$, this mapping must be a homomorphism, since we must have $\chi(T_1T_2)w=wT_1T_2=\chi(T_1)wT_2=\chi(T_1)\chi(T_2)w$. If w induces χ and $A \in G'$, then mapping

Aw induces the homomorphism $\chi' = A\chi A^{-1}$. If w induces f, then f is uniquely determined and is also induced by every mapping $Aw, A \in G'$.

C) A continuous mapping $w: U \to U$ induces a continuous mapping $f: U/G \to U/G'$ if and only if it induces a homomorphism $\chi: G \to G'$. Every continuous mapping $f: U/G \to U/G'$ is induced by a mapping $w: U \to U$, and w is uniquely determined by f, except that it may be replaced by $Aw, A \in G'$.

Proof. Assume that w is given and (6.2) holds. Then the equivalence class $[w(z)]_{G'}$ depends only on the equivalence class $[z]_G$ so that (6.1) defines a mapping $f: U/G \to U/G'$. One verifies at once that the latter mapping is continuous. Assume next that (6.1) holds for some continuous mapping $f: U/G \to U/G'$. For every $z \in U$ and $T \in G$ there is a $\chi(z,T) \in G'$ such that $w(T(z)) = \chi(z,T)(w(z))$. But $\chi(z,T)$ must depend continuously on z; since G' is discrete, $\chi(z,T) = \chi(T)$. That χ is a homomorphism follows from B).

Now let $f: U/G \to U/G'$ be a given continuous mapping. For every z_o in U there is an equivalence class $[\zeta_o]_{G'}$ such that $f(\Pi_G(z_o)) = [\zeta_o]_{G'}$. If we choose some element ζ_1 in $[\zeta_o]_{G'}$, then there exists a uniquely determined continuous function $w(z)$ defined in a neighborhood of z_o such that $w(z_o) = \zeta_1$ and (6.1) holds. Since U is simply connected, the definition of w can be uniquely continued over the whole of

U (monodromy theorem). The only free choice we had was that of the point ζ_1. Replacing ζ_1 by $\zeta_2 = A(\zeta_1)$, $A \in G'$, amounts to replacing w by Aw.

D) <u>Let</u> $w_o: U \to U$ <u>and</u> $w_1: U \to U$ <u>be continuous mappings which induce the homomorphisms</u> $\chi_o: G \to G'$ <u>and</u> $\chi_1: G \to G'$, <u>and the mappings</u> $f_o: U/G \to U/G'$ <u>and</u> $f_1: U/G \to U/G'$, <u>respectively. The mappings</u> f_o <u>and</u> f_1 <u>are homotopic if and only if</u> there is an $A \in G'$ such that $\chi_1 = A\chi_o A^{-1}$.

Proof (following Ahlfors). Assume first that $\chi_1 = A\chi_o A^{-1}$, $A \in G'$. Replacing w_1 by $A^{-1}w_1$ we may assume that $\chi_1 = \chi_o$. For every $t, 0 \leq t \leq 1$, let the point $w_t(z)$ divide the non-Euclidean segment $(w_o(z), w_1(z))$ in the ratio $t : 1 - t$. One verifies that $w_t(z)$ depends continuously on t and z, and that $w_t T = \chi_o(T) w_t$ for all $T \in G$. Hence (cf. C)) w_t induces a continuous mapping $g_t: U/G \to U/G'$. This mapping depends continuously on t and $g_o = f_o$, $g_1 = f_1$.

Assume now that the mappings f_o and f_1 are homotopic so that there exists a continuous mapping $g_t: U/G \to U/G'$ depending continuously on $t, 0 \leq t \leq 1$ with $g_o = f_o$, $g_1 = f_1$. Choose points z_o, ζ_o so that $f(\Pi_G(z_o)) = \Pi_{G'}(\zeta_o)$. There is a continuous complex-valued function $\delta(t)$, $0 \leq t \leq 1$, with $\delta(0) = \zeta_o$ and $g_t(\Pi_G(z_o)) = \Pi_{G'}(\delta(t))$. By the proof of C) there exists a continuous mapping $\hat{w}_t: U \to U$ satisfying the condition $w_t(z_o) = \delta(t)$ and inducing g_t. This mapping is easily seen to depend continuously on t. Let

$\hat{\chi}_t : G \to G'$ be the homomorphism induced by $\hat{w}_t$. For every $T \in G$ the element $\hat{\chi}_t(T) \in G'$ depends continuously on t ; since G' is discrete, $\hat{\chi}_t = \hat{\chi}_o$. By C) we have that $w_o = A_o\hat{w}_o$, $w_1 = A_1\hat{w}_1$, where $A_o, A_1 \in G'$. Hence $\chi_o = A_o\hat{\chi}_o A_o^{-1}$, $\chi_1 = A_1\hat{\chi}_o A_1^{-1}$, and setting $A = A_1A_o^{-1}$ we have $\chi_1 = A\chi_o A^{-1}$.

E) Let the continuous mapping $w : U \to U$ induce the continuous mapping $f : U/G \to U/G'$ and the homomorphism $\chi : G \to G'$. The homomorphisms of fundamental groups $f_{\Pi_G(z_o)}$ and $\Lambda_{G',w(z_o)}\chi\Lambda^{-1}_{G,z_o}$ are equivalent.

The proof follows simply by recalling the definitions of f_p and $\Lambda_{G,z}$ given in the previous section. In conjunction with C) and D) we obtain a proof of the topological theorem stated in A).

F) Under the hypothesis of E) the mapping f is a homeomorphism onto if and only if w is (that is, $w \in W$) and in this case χ is an isomorphism onto.

Assume that f is a homeomorphism and $f(U/G) = U/G'$. Then f_p is clearly an isomorphism onto and the assertion concerning χ follows from E). Suppose that $w(z_1) = w(z_2)$. Then $f(\Pi_G(z_1)) = f(\Pi_G(z_2))$ so that there is a $T \in G$ with $z_2 = T(z_1)$ and by (6.1) we have $w(z_2) = \chi(T)(w(z_1)) = w(z_1)$. Since the group G' is fixed-point-free, $\chi(T) = 1$ and also $T = 1$, $z_2 = z_1$. Thus w is one-to-one. Finally, $w(U) = U$; for, if w would omit a value $\zeta \in U$ it would also omit all

values in $[\zeta]_{G'}$ and f would omit the point $\Pi_{G'}(\zeta)$. It is plain that f is onto if w is and that w must fail to be one-to-one if f does.

We note that an orientation preserving homeomorphism f is induced by an element of W_o.

G) An element $w \in W$ will be said to be compatible with the Fuchsian group G if $wGw^{-1} \subset H$.

<u>If</u> $w \in W$ <u>is compatible with the Fuchsian group</u> G, <u>then</u> $G' = wGw^{-1}$ <u>is a Fuchsian group and</u> w <u>induces a homeomorphism of</u> U/G <u>onto</u> U/G'.

Proof. Every inner automorphism of W is a homeomorphism; hence G' is a discrete group of non-Euclidean motions. Assume that $A \in G'$ and $A(\zeta) = \zeta$ for some $\zeta \in U$. Then $T = w^{-1}Aw \in G$ and for $z = w^{-1}(\zeta)$ we have that $T(z) = z$. Hence $T = 1$, $A = 1$. Thus G' is fixed-point-free. Clearly w induces the isomorphism $\chi(T) = wTw^{-1}$ of G onto G'. The remaining statement follows from F).

7. Quasiconformal mappings of Riemann surfaces

In this section we define quasiconformality for mappings of Riemann surfaces and establish a one-to-one correspondence between quasiconfornal homeomorphisms of a Riemann surface and a class of differentials defined on the surface.

A) Let S and S' be Riemann surfaces and $f : S \rightarrow S'$ a

homeomorphism onto. Let z and ζ be local parameters defined in neighborhoods of points $p_o \varepsilon S$ and $q_o = f(p_o)$, respectively. Thus $z = g(p)$ and $\zeta = h(q)$ are conformal homeomorphisms of neighborhoods of the points p_o and q_o into the complex plane. Near p_o the mapping f can be written in the form $\zeta = w(z)$. More precisely: $w = hfg^{-1}$. If this mapping w is k quasiconformal (cf. §3, A)) we say that f is k quasiconformal in the neighborhood of p_o. The definition is obviously independent of the choice of local parameters (cf. §3, C)). If f is k quasiconformal in the neighborhood of every point of S, f is called quasiconformal, more precisely k quasiconformal. We denote by $k[f]$ the smallest number k' such that f is k' quasiconformal, and set $K[f] = (1 + k[f])/(1 - k[f])$. If f is not quasiconformal, we set $k[f] = 1$, $K[f] = \infty$.

It follows from §3, C) that $k[f^{-1}] = k[f]$ and that if $f_1 : S' \to S''$ is another homeomorphism, then

$$K[f_1 f] \leq K[f_1]K[f] . \tag{7.1}$$

1 quasiconformal mappings are conformal (cf. §2, C)).

B) A differential of type (m,n) on a Riemann surface S is a rule associating with every local parameter z defined on a domain $D \subset S$ a measurable function $\psi(z), z \varepsilon D$, in such a way that the expression $\psi(z)dz^m d\bar{z}^n$ is invariant under parameter changes. For a differential $\mu(z)d\bar{z}/dz$ of type $(-1,1)$ the absolute value of the coefficient, $|\mu(z)|$, is a scalar

function on S . If $|\mu| \leq k < 1$ we call $\mu(z)$ a Beltrami coefficient and $\mu d\bar{z}/dz$ a Beltrami differential.

<u>There exists a natural one-to-one correspondence between Beltrami differentials on S and quasiconformal homeomorphisms of S</u> .

Let $f : S \to S'$ be a given quasiconformal homeomorphism. Let z and $\zeta = w(z)$ have the same meaning as in A). The function $w(z)$ satisfies, in a neighborhood of $z_0 = g(p_0)$, a definite Beltrami equation $w_{\bar{z}} = \mu(z) w_z$ (cf. §4, D)). A direct computation (which is legitimate though we work with generalized derivatives) shows that $\mu d\bar{z}/dz$ is a differential of type (-1,1). We call it the Beltrami differential of the mapping f. Now let $\mu d\bar{z}/dz$ be a given Beltrami differential on S. We define on S a new conformal structure based on the conformal metric $|dz + \mu d\bar{z}|$ and call the new Riemann surface thus obtained S^μ. More precisely, the points and the open sets on S^μ are those of S and a local parameter on S^μ is a univalent function $\zeta = w(z)$ of a local parameter z of S which satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$. The legitimacy of this definition follows from the results stated in §4, B) and C). The identity mapping $f^\mu : S \to S^\mu$ is quasiconformal and its Beltrami coefficient is precisely μ ; we call f^μ the natural mapping induced by $\mu d\bar{z}/dz$. It is clear that every quasiconformal mapping may be considered as a natural mapping induced by its Beltrami differential.

We norm the space of Beltrami differentials on S by

setting $\| \mu d\bar{z}/dz \| = \| \mu \| =$ true maximum of $|\mu|$. Then the natural mapping induced by $\mu d\bar{z}/dz$ satisfies the equation $k[f^{\mu}] = \| \mu \|$.

C) Let G be a Fuchsian group. Since $z \in U$ is a local parameter in the neighborhood of every point of U/G we conclude (cf. §6, F)) that a homeomorphism $f : U/G \to U/G'$ is k quasiconformal if and only if it is induced by a k quasiconformal homeomorphism $w : U \to U$.

Every Beltrami differential on U/G may be written in the form $\mu(z)d\bar{z}/dz$ where $\mu(z)$ is a bounded measurable function defined in U and such that $|\mu| \leq k < 1$ and

$$\mu(T(z)) = \mu(z)T'(z)/\overline{T'(z)} \quad \text{for } T \in G . \tag{7.2}$$

We call such functions Beltrami coefficients compatible with G. For every Beltrami coefficient μ compatible with G, $\mu\, d\bar{z}/dz$ is a Beltrami differential on U/G.

D) <u>Let G be a Fuchsian group and $\mu(z)$ a Beltrami coefficient compatible with G. There exists a solution w of the Beltrami equation</u>

$$w_{\bar{z}} = \mu(z)w_z$$

<u>which maps U topologically onto itself, and $w \in W_0 \cap W^*$. Moreover, w is compatible with G (cf. §6, G)), $U/(wGw^{-1}) = (U/G)^{\mu}$, and w induces the natural mapping f^{μ} of U/G onto $(U/G)^{\mu}$.</u>

Proof. Since the upper half-plane is conformally equivalent with the unit disc, the equivalence being established,

say by the mapping Q (§5, A)), the existence of w follows from §4, B) and C). All solutions of Beltrami equations being orientation preserving, $w \in W_0$. That $w \in W^*$ follows from §3, E). Now set, for some $T \in G$, $\hat{w}(z) = w(T(z))$. A direct computation based on (7.2) shows that $\hat{w}_{\bar{z}} = \mu(z)\hat{w}_z$. (This computation is legitimate, though we work with generalized derivatives.) By §4, B) we conclude that there exists an analytic function A(w) such that $\hat{w}(z) = w(T(z)) = A(w(z))$. But $A = wTw^{-1} \in W_0$, so that A is a non-Euclidean motion. Hence $wGw^{-1} \subset H$ and w is compatible with G. It follows that w induces a homeomorphism of U/G onto $U/(wGw^{-1})$. The proof of the remaining statements is obvious.

We remark that w is not uniquely determined by μ but may be replaced by $w_1 = Aw, A \in H$ (cf. §6, C)). But $w_1Gw_1^{-1} = A(wGw^{-1})A^{-1}$ so that the Riemann surfaces $U/(wGw^{-1})$ and $U/(w_1Gw_1^{-1})$ are conformally equivalent.

8. Teichmüller mappings

We shall define Teichmüller mappings using everywhere regular quadratic differentials since such mappings turn out to be extremal in the problems considered here. Mappings defined by meromorphic quadratic differentials with only simple poles are extremal in the class of mappings attaining preassigned values at given points. It is not known what extremal property, if any, is enjoyed by mappings defined by

meromorphic quadratic differentials with poles of higher order.

A) A quadratic differential on a Riemann surface S is a differential $\phi(z)dz^2$ of type (2,0); it is called regular if the coefficient $\phi(z)$ is an analytic function of the local parameter z. Let such a differential be given, and assume that it does not vanish identically. At a point p of S we have then $\phi dz^2 = z^m(a_o + a_1 z + \ldots)dz^2$ where z is a local parameter which vanishes at p, and $a_o \neq 0$. The non-negative integer $m = m(p)$ is called the order of the differential at p; if $m > 0$, p is called a zero of multiplicity m. Set

$$(8.1) \qquad \zeta = \zeta(z) = \left\{ \int_o^z \phi(z)^{1/2} dz \right\}^{2/(m+2)} .$$

Then ζ is a local parameter defined near p; we call it the natural parameter belonging to ϕdz^2 at p. The natural parameter is characterized by the relations

$$(8.2) \quad \zeta = 0 \text{ at } p, \phi dz^2 = d\zeta^2 \text{ if } m = 0, \phi dz^2 = \left(\frac{m+2}{2}\right)^2 \zeta^m d\zeta^2 \text{ if } m > 0 .$$

It may be multiplied by an $(m + 2)$nd root of unity, but is otherwise uniquely determined.

B) Let G be a Fuchsian group and $S = U/G$. Since $z \in U$ is a local parameter in the neighborhood of every point on S, every regular quadratic differential on S may be written as $\phi(z)dz^2$ where $\phi(z)$ is holomorphic in U and satisfies the functional equation

$$\phi(T(z)) = \frac{\phi(z)}{T'(z)^2} \quad \text{for all} \quad T \in G \;. \tag{8.3}$$

Conversly, if $\phi(z)$ is a holomorphic solution of (8.3) defined in U, then ϕdz^2 is a regular quadratic differential on S.

C) Let f be a homeomorphism of a Riemann surface S onto another such surface S'. We say that f is a Teichmüller mapping if f is conformal or if f is quasiconformal with a Beltrami coefficient (cf. §7, B)) of the form

$$\mu(z) = k \frac{\phi(z)}{|\phi(z)|} \tag{8.4}$$

where $\phi dz^2 \not\equiv 0$ is a regular quadratic differential on S and k a constant such that $0 < k < 1$. In the latter case we say that f is defined by ϕdz^2 and k. It is easy to see that in this case f determines uniquely the differential ϕdz^2, except for a positive constant factor, and that $K[f] = (1+k)/(1-k)$.

We remark that whenever $\phi(z)dz^2$ is a quadratic differential (regular or not) on S and $0 < k < 1$, (8.4) is a Beltrami coefficient on S.

D) <u>Let $f : S \to S'$ be a Teichmüller mapping defined by the regular quadratic differential $\phi(z)dz^2 \not\equiv 0$ on S and the constant k. Then there exists a uniquely determined regular quadratic differential $\psi(z')dz'^2$ on S' having the following properties. (i) The order of $\psi dz'^2$ at f(p) is equal to the order of ϕdz^2 at p. (ii) Let ζ be the natural parameter belonging to ϕdz^2 at a point $p \in S$ at which ϕdz^2</u>

has order zero. In the neighborhood of p *the mapping* f *may be described by the equation*

$$(8.5) \qquad \zeta' = \frac{\zeta + k\bar{\zeta}}{1 - k}$$

where ζ' *is the natural parameter belonging to* $\psi dz'^2$ *at* f(p).
(iii) *The inverse mapping* $f^{-1}: S' \to S$ *is a* Teichmüller *mapping defined by the differential* $-\psi dz'^2$ *and the constant* k.

Before proving this statement we remark that, setting $\zeta = \xi + i\eta$, $\zeta' = \xi' + i\eta'$, relation (8.5) may be written in the form

$$(8.6) \qquad \xi' = K\xi, \eta' = \eta \qquad (K = K[f]) .$$

Thus a Teichmüller mapping is, in the neighborhood of every point which is not a zero of the defining quadratic differential, a conformal transformation, followed by a fixed affine transformation, followed by another conformal transformation. This itself already suggests the extremal nature of Teichmüller mappings, if one thinks of the simple case of mapping one topological rectangle onto another (cf. §3, B)).

E) In order to prove D) we consider some point $p \in S$ at which ϕdz^2 has order m. Let ζ be a natural parameter belonging to ϕdz^2 at p and set

$$(8.7) \qquad \zeta' = \left(\frac{\zeta^{(m+2)/2} + k\bar{\zeta}^{(m+2)/2}}{1 - k} \right)^{2/(m+2)}$$
$$= (1-k)^{-2/(m+2)} \zeta \sum_{j=0}^{\infty} \binom{2/(m+2)}{j} k^j \left(\frac{\bar{\zeta}}{\zeta}\right)^{j(m+2)/2} .$$

If $m = o$ this is relation (8,5).

Now let z' be some local parameter on S' defined near the point $f(p)$. In the neighborhood of p the mapping f is described by a function $z' = w(\zeta)$ which is a homeomorphic solution of the Beltrami equation

$$(8.8) \qquad w_{\bar{\zeta}} = k\left(\frac{\zeta}{|\zeta|}\right)^m w_\zeta \, .$$

But the function ζ' defined above is, for small values of $|\zeta|$, also a homeomorphic solution of this Beltrami equation. Thus ζ is, by §4, B), an analytic function of $z' = w(\zeta)$ and hence a local parameter on S near $f(p)$. We call it, for the time being, the distinguished parameter at $f(p)$.

Now let $p_1 \neq p$ be a point sufficiently close to p and t the value of ζ at p_1. Since the zeros of a regular quadratic differential are isolated, the order of ϕdz^2 at p_1 is 0 and, by A), a natural parameter at p_1 is

$$\zeta_1 = \int_t^\zeta \left(\frac{m+2}{2}\right) \zeta^{m/2} d\zeta = \zeta^{(m+2)/2} - t^{(m+2)/2} \, .$$

Hence
$$\zeta_1' = \frac{\zeta_1 - k\bar{\zeta}_1}{1 - k}$$

is the distinguished parameter on S' at $f(p_1)$, and comparing this with (8,7) we see that $4d\zeta_1'^2 = (m+2)^2 \zeta'^m d\zeta'^2$. This means that we may define a regular quadratic differential on S' by setting

$$\psi(z')dz'^2 = \left(\frac{m+2}{2}\right)^2 \zeta'^m d\zeta'^2 \, .$$

This differential has properties (i) and (ii) by construction. Solving (8.7) for ζ we see that ζ as a function of ζ' satisfies the Beltrami equation

$$\zeta_{\bar{\zeta}} = - k \left(\frac{\bar{\zeta'}}{|\zeta'|}\right)^m \zeta_{\zeta'}$$

Consequently $\psi dx'^2$ also has property (iii).

9. Marked Riemann Surfaces

In this section we define marked Riemann surfaces, a concept which is identical in content with Teichmüller's topologically determined Riemann surfaces.

A) Let S be a closed Riemann surface of genus $g > 1$. A standard set of generators at a point $p \in S$ is an ordered set $a = (a_1, a_2, \ldots, a_{2g})$ of elements of the fundamental group $\pi_1(S,p)$ which generate this group with the single defining relation

$$\prod_{j=1}^{g} (a_{2j} a_{2j-1}^{-1} a_{2j}^{-1} a_{2j-1}) = 1 .$$

The existence of a standard set is classical; the acutal construction of such a set involves the so called canonical dissection of the surface.

Two standard sets of generators, a at p and b at q, will be called equivalent if there exists an allowable isomorphism

(cf. §5, E)) $\omega:\pi_1(S,p) \to \pi_1(S,q)$ such that $\omega(a) = b$, that is $\omega(a_j) = b_j$, $j = 1,2,\ldots,2g$. A pair consisting of a Riemann surface S together with an equivalence class α of standard sets of generators will be called a marked Riemann surface and will be denoted by (S,α).

An equivalent definition reads as follows. A marked Riemann surface is a closed Riemann surface S of genus $g > 1$ together with a canonical dissection of S; two dissections define the same marked surface, if one is obtained from the other by a homeomorphism of S onto itself which is homotopic to the identity.

B) Let (S,α) and $S',\alpha')$ be two marked Riemann surfaces of the same genus. A homeomorphism $f : S \to S'$ will be called a mapping of (S,α) onto (S',α') and will be denoted by $f : (S,\alpha) \to (S',\alpha')$ if for a point $p \in S$ the induced isomorphism $f_p : \pi_1(S,p) \to \pi_1(S',f(p))$ takes a set of generators belonging to α into a set of generators belonging to α'.

<u>If</u> $f : (S,\alpha) \to (S',\alpha')$ <u>and</u> $h : (S,\alpha) \to (S',\alpha')$, <u>then the mappings</u> f <u>and</u> h <u>are homotopic</u>. <u>Conversely, if</u> $h : S \to S'$ <u>is a homeomorphism homotopic to</u> f, <u>then</u> $h : (S,\alpha) \to (S',\alpha')$.

The proof follows at once from §6, A).

C) <u>If</u> (S,α) <u>and</u> (S',α') <u>are two marked Riemann surfaces of the same genus, then there exists a homeomorphism</u> $f : S \to S'$ such that $f: (S,\alpha) \to (S',\alpha')$.

This follows from the known topological theorem (Dehn) which asserts that every automorphism of the fundamental group of a closed surface can be realized by a homeomorphism of the surface onto itself. For the proof we refer to the paper of Mangler [13] and the references given there.

The homeomorphism f is, of course, determined only up to a homotopy. If f is orientation preserving (orientation reversing), we say that the marked Riemann surfaces (S,α) and (S',α') are similarly orientated (have opposite orientations).

D) If (S,α) and (S',α') are two similarly oriented marked Riemann surfaces of the same genus, then there exists a quasiconformal mapping $f : (S,\alpha) \longrightarrow (S',\alpha')$.

Had we demanded that the homeomorphism f be continuously differentiable everywhere the proof would be somewhat laborious. Since we use a very general definition of quasiconformality, the proof presents no difficulties and may be omitted.

The assertion means that every homeomorphism $f:S \longrightarrow S'$ is homotpic to a quasiconformal mapping. The corresponding assertion for open surfaces is trivially false. For instance, there is no quasiconformal mapping of the finite plane onto the unit disc.

10. Statement of Teichmüller's Theorem

For the sake of convenience, we separate Teichmüller's theorem into three assertions.

A) Theorem I. Let $f_o: (S,\alpha) \to (S',\alpha')$ be a Teichmüller mapping of one marked Riemann surface of genus $g > 1$ onto another such surface. If $f : (S,\alpha) \to (S',\alpha')$ is a mapping distinct from f then $K[f_o] < K[f]$.

The theorem states that for a closed Riemann surface of genus $g > 1$ a Teichmüller mapping has strictly smaller dilitation than any homeomorphism homotopic to it. As a corollary we obtain that no two distinct Teichmüller mappings are homotopic.

B) Theorem II. Let (S,α) and (S',α') be two similarly oriented marked Riemann surfaces of the same genus $g > 1$. Then there exists a Teichmüller mapping $f_o: (S,\alpha) \to (S',\alpha')$.

In other words, every homotopy class of an orientation preserving homeomorphism $f : S \to S'$ contains a Teichmüller mapping. By the uniqueness theorem (Theorem I) this mapping is extremal.

C) The Teichmüller distance between two marked similarly oriented Riemann surfaces of genus $g > 1$ is, by definition,

$$(10.1) \qquad d((S,\alpha),(S',\alpha')) = \log K[f_o]$$

where $f_o: (S,\alpha) \rightarrow (S',\alpha')$ is the Teichmüller mapping which exists and is unique by virtue of I and II. An equivalent definition is

$$(10.2) \qquad d((S,\alpha),(S',\alpha')) = \inf \log K[f]$$

taken over all mappings $f : (S,\alpha) \rightarrow (S',\alpha')$. The fact that the distance between two distinct marked Riemann surfaces is positive and that the distance function is symmetric is an immediate consequence of the definition. To prove the triangle inequality consider a Teichmüller mapping $f_1: (S',\alpha') \rightarrow (S'',\alpha'')$. Using definition (10.2) and inequality (7.1), we conclude that the distance between (S,α) and (S'',α'') does not exceed $\log K[f_o] + \log K[f_1]$. Thus similarly oriented marked Riemann surfaces of genus $g > 1$ form a metric space R_g.

__Theorem__ III. _The metric space R_g is homeomorphic to the Euclidean space of $6g - 6$ dimensions._

This is a partial solution of the so-called problem of moduli.

11. The metric induced by a regular quadratic differential

This and the following section contain an outline of the proof of the uniqueness theorem (Theorem I). We merely reword Teichmüller's own argument, making sure that it applies

also to the general definition of quasiconformality used here. (Another modification of Teichmüller's uniqueness proof was given by Ahlfors.)

A) Let $\phi(z)dz^2$ be a regular quadratic differential defined on a closed Riemann surface S. We define on S a metric based on the line element $ds_\phi = |\phi(z)|^{1/2}|dz|$; the corresponding area element is $dA_\phi = |\phi(z)|dxdy$. The length of a curve α in this metric will be denoted by $|\alpha|_\phi$.

At points distinct from the zeros of the differential the metric is locally Euclidean; the zeros are singular points. A geodesic is characterized by the property that along it $\arg \phi(z)dz^2$ is constant. A geodesic which passes through a zero of order m may possess there a corner with an angle not less that $2\pi/(m+2)$. One verifies this by using natural parameters. Hence every geodesic can be continued indefinitely and to an arbitrarily given length, though the continuation through a zero is not unique. One now concludes, in the usual way, that if α is a curve on S with endpoints p and q, then the homotopy class of α relative to the endpoints, $[\alpha]_{p,q}$, contains a geodesic (which minimizes the ϕ-length). Furthermore, the geodesic is unique. This can be proved by repeating the classical argument for a surface with a smooth Riemannian metric of non-positive curvature. In fact, the metric ds_ϕ has a non-positive curvature in the sense that the sum of angles in a geodesic triangle does not exceed π.

B) A geodesic arc along which $\phi dz^2 > 0$ (or $\phi dz^2 < 0$) is called horizontal (vertical). A geodesic arc which does not pass through a zero of ϕdz^2 is called free. A free horizontal arc is the unique shortest curve in its homotopy class.

Let p_0 be a point which is not a zero of ϕdz^2, and let p_0 be the midpoint of a free vertical arc β of length $2b$. Assume further that every point of β is a midpoint of a free horizontal arc of length $2a$. Then these horizontal arcs cover a simply connected domain $\triangle(p_0,a,b)$ on the universal covering surface of S and the natural parameter $\zeta(p) = \xi + i\eta$ belonging to ϕdz^2 at p_0 maps $\triangle(p_0,a,b)$ conformally onto the rectangle $-a < \xi < a, -b < \eta < b$. This construction makes it clear what we mean by saying that some property holds for almost all horizontal arcs.

For $c > 0$ let Ω_c denote the set of those points of S which can be joined to a zero of ϕdz^2 by a horizontal arc of length not exceeding c. Since only a finite number of horizontal arcs emerge from each of the isolated zeros of ϕdz^2, the set Ω_c is the union of at most countably many analytic arcs (the union of finitely many analytic arcs if S is closed). The union Ω_∞ of all Ω_c has two-dimensional measure zero.

If p_0 is a point in the complement of Ω_c and $0 < a < c$, then there is a $b > 0$ such that we may form the domain $\triangle(p_0,a,b)$.

C) Lemma A. Let $\phi(z)dz^2 \not\equiv 0$ be a regular quadratic differential on a closed Riemann surface and let $f : S \to S$ be

<u>a homeomorphism homotopic to the identity. There exists a constant</u> $A > 0$ <u>such that for every free horizontal arc</u> α

$$|f(\alpha)|_\phi \geq |\alpha|_\phi - 2A \ . \tag{11.1}$$

Proof. By assumption there exists a continuous mapping $f_t: S \longrightarrow S$ depending continuously on $t, 0 \leq t \leq 1$, with $f_1 = f$ and $f_o(p) = p$. For every $p_o \in S$ let $\gamma(p_o)$ denote the curve $p = f_t(p_o), 0 \leq t \leq 1$. Let $\delta(p_o)$ be the unique geodesic in the homotopy class of $\gamma(p_o)$ and set

$$A = \sup |\delta(p)|_\phi, p \in S \ . \tag{11.2}$$

Compactness of S implies easily that $A < \infty$. Now let α be a free horizontal arc with endpoints p and q. The curve $\gamma(p)f(\alpha)\gamma(q)^{-1}$ has the same endpoints and is homotopic to α. The same is ture of the curve $\delta(p)f(\alpha)\delta(q)^{-1}$. Thus $|\alpha|_\phi \leq |\delta(p)|_\phi + |f(\alpha)|_\phi + |\delta(q)|_\phi$ and comparing this inequality with (11.2) we obtain (11.1).

D) Let $\phi(z)dz^2 \not\equiv 0$ be a regular quadratic differential on a Riemann surface S, $\psi(z')dz'^2$ another such differential on a Riemann surface S' and f a quasiconformal homeomorphism of S onto S'. On S we define the measurable non-negative point-functions

$$\lambda_{f;\ \phi,\psi}(p) = |\phi(z)^{-1/2}w_z + \overline{\phi(z)}^{-1/2}w_{\bar{z}}|\ |\psi(w(z))|^{1/2} \ , \tag{11.3}$$

$$j_{f;\ \phi,\psi}(p) = |\phi(z)|^{-1}(|w_z|^2 - |w_{\bar{z}}|^2)|\psi(w(z))| \ . \tag{11.4}$$

Here z is some local parameter defined near $p_o \varepsilon S$, w some local parameter defined near $f(p_o) \varepsilon S'$, and $w = w(z)$ the function defined by the mapping f. If p_o and $f(p_o)$ are not zeros of the corresponding quadratic differentials we have, in terms of natural parameters $\zeta = \xi + i\eta$, $\zeta' = \xi' + i\eta'$,

$$(11.3') \qquad \lambda_{f\,;\,\phi,\psi} = [(\xi'_\xi)^2 + (\eta'_\xi)^2]^{1/2} ,$$

$$(11.4') \qquad j_{f\,;\,\phi,\psi} = \xi'_\xi \eta'_\eta - \xi'_\eta \eta'_\xi .$$

This shows the simple geometric meaning of the functions defined. If $G \subset S$ is an open set, then

$$(11.5) \qquad \iint_G j_{f\,;\,\phi,\psi} dA_\phi = \iint_{f(G)} dA_\psi ,$$

and for almost all horizontal arcs α on S we have

$$(11.6) \qquad \int_\alpha \lambda_{f\,;\,\phi,\psi} ds_\phi = \int_{f(\alpha)} ds_\psi = |f(\alpha)|_\psi .$$

We also note the inequality

$$(11.7) \qquad \lambda^2_{f\,;\,\phi,\psi} \leq K[f] j_{f\,;\,\phi,\psi}$$

which holds almost everywhere and is an immediate consequence of the definition of quasiconformality.

E) The uniqueness proof is based on the following

Lemma B. Let $\phi(z)dz^2 \not\equiv 0$ be a regular quadratic differential on a closed Riemann surface S and $f : S \rightarrow S$ a homeomorphism homotopic to the identity. Then

(11.8) $$\iint_S \lambda_{f;\phi,\phi} dA_\phi \geq \iint_S dA_\phi .$$

Proof. Let $\nu(p)$ be any non-negative measurable integrable function defined on S. For some $a > 0$ define a new function $\tilde{\nu}(p;a)$ by setting

$$\tilde{\nu}(p,a) = \int_\alpha \nu(p) ds_\phi$$

if p is the midpoint of a free horizontal arc α with $|\alpha|_\phi = 2a$ and $\nu(p)$ is measurable and integrable on α, and $\tilde{\nu}(p,a) = 0$ otherwise. A simple argument, given below, shows that

(11.9) $$\iint_S \tilde{\nu}(p,a) dA_\phi = 2a \iint_S \nu(p) dA_\phi .$$

We apply this relation to the function $\lambda_{f;\phi,\phi}$. Noting (11.6) and Lemma A we have that $\tilde{\lambda}_{f;\phi,\phi}(p,a) \geq 2(a-A)$ almost everywhere. Thus

$$2a \iint_S \lambda_{f;\phi,\phi} dA_\phi \geq 2(a-A) \iint_S dA_\phi ,$$

and since this must hold for all $a > 0$, and A is fixed, inequality (11.8) follows.

F) It remains to verify relation (11.9). It is clear that we may assume that the function ν is continuous and vanishes in a neighborhood of Ω_{4a} (cf. B)), since then the general case can be treated by an obvious limiting process. Furthermore, using a partition of unity, we may assume that

ν has arbitrarily small compact support in $S - \Omega_{4a}$. More precisely, we assume that we are given a and $b, 0 < b < a$ and a point $p_o \in S$ such that we may form the domain $\triangle(p_o, 3b, 3a)$ described in B), and that every point of the support of ν has a ϕ-distance from p_o not exceeding b. Introducing the natural parameter $\zeta(p) = \xi + i\eta$ belonging to ϕdz^2 at p_o and setting $\nu(\zeta(p)) = n(\zeta)$, we see that the relation to be proved becomes

$$\iint_{-\infty}^{+\infty} \left\{ \int_{-a}^{a} n(\xi+\tau, \eta) d\tau \right\} d\xi d\eta = 2a \iint_{-\infty}^{+\infty} n(\xi, \eta) d\xi d\eta .$$

Its validity is, therefore, obvious.

12. Uniqueness proof

We are now in a position to prove Theorem I.

A) Let S be a closed Riemann surface of genus $g > 1$. If $f : S \rightarrow S$ is a conformal homeomorphism homotopic to the identity, then f is the identity.

This classical statement follows at once from the considerations of §6. Set $S = U/G$ and let $w : U \rightarrow U$ be the homeomorphism inducing f. If f is conformal, w must be conformal, that is an element of H. If f is homotopic to the identity, w induces the identity isomorphism on G. Hence w commutes with every element of G, and, since G must contain

two non-commuting non-Euclidean translations, w is the identity and so is f.

This argument proves Theorem I for the case $K[f_o] = 1$.

B) Now let $f_o: S \longrightarrow S'$ be a Teichmüller mapping between two closed Riemann surfaces of genus $g > 1$, defined by the regular quadratic differential $\phi(z)dz^2 \not\equiv 0$ on S and some constant. Let $\psi(z')dz'^2$ be the regular quadratic differential on S' constructed in §8, E). Finally let $f : S \longrightarrow S'$ be a quasiconformal mapping homotopic to f_o. We set $K[f_o] = K_o, K[f] = K$ and show first that

$$K \geq K_o \tag{12.1}$$

The mapping $h = f_o f^{-1}$ is a quasiconformal homeomorphism of S' onto itself which is homotopic to the identity, so that by Lemma B of the preceding section and by Schwarz's inequality

$$\iint_{S'} dA_\psi \leq \iint_{S'} \lambda^2_{h;\psi,\psi} dA_\psi \tag{12.2}$$

On the other hand, it follows from §8, E) and the definitions given in §11, D) that we have

$$\lambda_{f;\phi,\psi}(p) = K_o \lambda_{h;\psi,\psi}(f(p)) , \tag{12.3}$$

$$j_{f_o;\phi,\psi} = K_o \tag{12.4}$$

Noting property (11.5) of the function j and using relations (12.3), (12.4) together with inequality (11.7) we obtain

$$\iint_{S'} \lambda^2_{h;\psi,\psi} dA_\psi = K_o^{-2} \iint_{S'} \lambda^2_{f;\phi,\psi} dA_\psi = K_o^{-1} \iint_{S} \lambda_{f;\phi,\psi} dA_\phi$$

$$(12.5) \qquad \leq (K/K_o) \iint_S j_{f;\phi,\psi} dA_\phi = (K/K_o) \iint_{S'} dA_\psi .$$

This inequality, together with (12.2), implies (12.1).

C) Assume now that the equality sign holds in (12.1). Then it must hold also in (12.5), so that be (11.7) we must have

$$(12.6) \qquad \lambda^2_{f;\phi,\psi} = K_o j_{f;\phi,\psi} .$$

Let $p_o \in S$ be a point such that $\phi dz^2 \neq 0$ at p_o and $\psi dz'^2 \neq 0$ at $f(p_o)$. Let $\zeta = \xi + i\eta$ be the natural parameter belonging to ϕdz^2 at p_o, and $\zeta' = \xi' + i\eta'$ and $\zeta^o = \xi^o + i\eta^o$ the natural parameters belonging to $\psi dz'^2$ at $f(p_o)$ and at $f_o(p_o)$, respectively. By §8, D), or by the definition of a Teichmüller mapping, we see that the function $\zeta^o(\zeta)$ defined by the mapping f_o satisfies the Beltrami equation

$$\zeta^o_{\bar\zeta} = k_o \zeta^o{}_\zeta \quad \left(k_o = \frac{K_o - 1}{K_o + 1} \right) .$$

On the other hand, by (12.6) and by inequality (3.1') we have that the function $\zeta'(\zeta_o)$ defined by the mapping f satisfies the relations

$$(\xi'_\xi)^2 + (\eta'_\xi)^2 = K_o(\xi'_\xi\eta'_\eta - \xi'_\eta\eta'_\xi) ,$$

$$(\xi'_\xi)^2 + (\xi'_\eta)^2 + (\eta'_\xi)^2 + (\eta'_\eta)^2 \leq [K_o + (1/K_o)](\xi'_\xi\eta'_\eta - \xi'_\eta\eta'_\xi)$$

A simple computation shows that these two relations imply that

$$\zeta'_{\bar{\zeta}} = k_o\zeta'_\zeta \, .$$

Noting §4, B) we conclude that the function $\zeta'(\zeta_o)$ defined by the mapping h is analytic.

Thus the mapping h is conformal at all points of S, except perhaps at zeros of $\phi dz'^2$ or at preimages, under f, of zeros of $\psi dz'^2$. By the theorem on removable singularities it is conformal also at these points. Since $h : S' \to S'$ is homotopic to the identity, it is the identity (cf. A)). Thus $f = f_o$ and Theorem I is proved.

13. Moduli of marked Riemann surfaces

In this section we prepare for the existence proof by associating with each marked Riemann surface of genus $g > 1$ a set of $6g - 6$ real numbers. These "moduli" are less symmetrical than the ones used by Fricke [6] but have the advantage of representing a marked Riemann surface as a point in a Euclidean space of the right dimension.

A) Let $g > 1$ be a given integer. An ordered set $(A_1,\ldots,A_{2g})$ of non-Euclidean translations will be called

normalized if the following conditions are satisfied. (i) The group G generated by $(A_1,\dots,A_{2g})$ is a Fuchsian group with a compact fundamental region. (ii) The transformations $A_1,\dots,A_{2g}$ form a set of generators with the single defining relation

$$\prod_{j=1}^{g}(A_{2j}A_{2j-1}^{-1}A_{2j}^{-1}A_{2j-1}) = 1 \ . \tag{13.1}$$

(iii) The repelling and attracting fixed points of A_{2g} are 0 and ∞, respectively, and the fixed points of A_{2g-1} are two real numbers the product of which has absolute value 1.

We shall denote the repelling and attracting fixed points and the invariants of A_j by σ_j, τ_j and λ_j, respectively. Condition (iii) means therefore that

$$\sigma_{2g} = 0, \tau_{2g} = \infty \ , |\sigma_{2g-1}\tau_{2g-1}| = 1 \ . \tag{13.2}$$

Actually we will always have that $\sigma_{2g-1}\tau_{2g-1} = -1$, but this is of no importance for what follows.

The numbers $\sigma_1, \tau_1, \lambda_1, \sigma_2, \dots, \lambda_{2g-2}$ will be called the coordinates of the normalized set $(A_1,\dots,A_{2g})$.

B) <u>A normalized set of non-Euclidean translations is uniquely determined by its coordinates.</u>

Proof. The coordinates determine the transformations $A_1,\dots,A_{2g-2}$ and hence, by (13.1), also the commutator $A_{2g}A_{2g-1}^{-1}A_{2g}^{-1}A_{2g-1} = B$. In view of (13.2) we may write

$$A_{2g-1}(z) = \frac{\alpha z+\beta}{\varepsilon\beta z+\delta}, A_{2g}(z) = \lambda z$$

where

(13.3) $\lambda > 1$; $\varepsilon = \pm 1$; α,β,δ are real, $\alpha\delta - \varepsilon\beta^2 = 1$, $\beta > 0$.

Thus

(13.4) $$B(z) = \frac{(\lambda\alpha\delta - \lambda^2\varepsilon\,\beta^2)z + (\lambda - \lambda^2)\beta\delta}{(\lambda - 1)\varepsilon\,\alpha\beta z + \lambda\alpha\delta - \varepsilon\beta^2}$$

and an easy computation shows that knowing $B(z)$ we can find numbers $\lambda,\varepsilon,\alpha,\beta,\delta$ satisfying (13.3), (13.4) in at most one way.

C) Using the same notations as in A) set $S = U/G$. Then S is a closed Riemann surface of genus g. For every $z \in U$ the natural isomorphism $\Lambda_{G,z}$ (cf. §5, E)) maps the normalized set $(A_1,\dots,A_{2g})$ onto a standard set of generators of the fundamental group $\pi_1(S,\Pi_G(z))$ and all standard sets so obtained are equivalent (cf. §9, A)). Denote their equivalence class by α. The marked Riemann surface (S,α) will be said to be defined by the normalized set $(A_1,\dots,A_{2g})$. The point $(\sigma_1,\tau_1,\lambda_1,\dots,\sigma_{2g-2})$ in the $(6g-6)$-dimensional Euclidean space will be called the representative point of (S,α).

D) <u>Every marked Riemann surface</u> (S,α) <u>of genus</u> $g > 1$ <u>is defined by a uniquely determined normalized set of non-Euclidean translations</u> (and thus has a representative point).

To prove this, represent S as U/G. The representation

is unique, except that we may replace G by $CGC^{-1}, C \in H$.
For some $z \in U$ set $\hat{A}_j = \Lambda^{-1}_{G,z}(a_j), j = 1,2,\ldots,2g$, where $a = (a_1,\ldots,a_{2g})$ is some standard set in α at $\Pi_G(z)$. Since the group G contains only the identity and non-Euclidean translations (cf. §5, D)) the $\hat{A}_j$ are non-Euclidean translations. For the same reason any two of them have four distinct fixed points, for otherwise their commutator would not have two distinct fixed points. It is clear that the set $(\hat{A}_1,\ldots,\hat{A}_{2g})$ has properties (i) and (ii). Denote the fixed points of $\hat{A}_j$ by $\hat{\sigma}_j, \hat{\tau}_j$. There exists an element $C \in H$ which satisfies the conditions
$C(\hat{\sigma}_{2g}) = 0, D(\hat{\tau}_{2g}) = \infty, |C(\sigma_{2g-1})C(\tau_{2g-1})| = 1$. Set
$A_j = C\hat{A}_jC^{-1}$. Then the set $(A_1,\ldots,A_{2g})$ has properties (i) - (iii) and defines (S,α). It is clear that $(A_1,\ldots,A_{2g})$ is uniquely determined by (S,α).

E) Now let $(A_1,\ldots,A_{2g})$ and $(A'_1,\ldots,A'_{2g})$ be two normalized sets of non-Euclidean translations which generate the groups G and G' and the marked Riemann surfaces (S,α) and (S',α'), respectively. Also, let w be a homeomorphism of U onto itself which satisfies the condition

$$A'_j = wA_jw^{-1}, j = 1,2,\ldots,2g \ . \tag{13.5}$$

Then $wGw^{-1} = G'$, so that w is compatible with G (cf. §6, G)) and therefore induces a homeomorphism $f : S \longrightarrow S'$. Noting §6, E) we conclude that $f : (S,\alpha) \longrightarrow (S',\alpha')$. This remark

will be used in the existence proof given below.

14. Existence proof

We are now in a position to prove Theorem II. The proof hinges on the Riemann-Roch theorem, or rather on a corollary concerning regular quadratic differentials.

A) Throughout this section we denote by (S^o, α^o) a fixed marked Riemann surface of genus $g > 1$. This marked surface is defined by a normalized set of non-Euclidean translations $(A_1^o, \ldots, A_{2g}^o)$. We denote by $\sigma_1^o, \tau_1^o, \ldots, \sigma_{2g}^o, \tau_{2g}^o$ the fixed points of these translations; they satisfy, of course, the normalization condition (13.2). G^o denotes the group generated by $(A_1^o, \ldots, A_{2g}^o)$.

We denote by M the set of all Beltrami coefficients $\mu(z), z \in U$, compatible with G_o (cf. §7, C)). The set is normed by setting $\|\mu\|$ = true maximum of $|\mu(z)|$, and a convergence concept is defined by requiring $\mu_n \to \mu$ to mean that $\|\mu_n\| \leq k < 1, n = 1, 2, \ldots,$ and $\mu_n(z) \to \mu(z)$ almost everywhere.

We denote by C the set of all representative points of marked Riemann surfaces of genus g which are oriented similarly to (S^o, α^o), and by C^* the set of the representative points of marked Riemann surfaces with opposite orientation. C and C^* are disjoint subsets of Σ_{6g-6} (the (6g-6) dimensional

Euclidean space); their union is the set of representative points of all marked Riemann surfaces of genus g.

If ξ is a point in a Euclidean space, $|\xi|$ denotes its distance from the origin.

B) For $\mu \in M$ let $w^{\mu}(z)$ denote the uniquely determined solution of the Beltrami equation $w_{\bar{z}} = \mu(z)w_z$ which maps U homeomorphically onto itself and satisfies the conditions

$$(14.1) \quad w^{\mu}(0) = 0, w^{\mu}(\infty) = \infty, |w^{\mu}(\sigma^{o}_{2g-1})w^{\mu}(\tau^{o}_{2g-1})| = 1$$

(cf. §7, D)). Set $A^{\mu}_j = (w^{\mu})A^{o}_j(w^{\mu})^{-1}, j = 1,2,\ldots,2g$. It is seen at once that $(A^{\mu}_1,\ldots,A^{\mu}_{2g})$ is a normalized set of non-Euclidean translations, for the fixed points of A^{μ}_j are $w^{\mu}(\sigma^{o}_j)$ and $w^{\mu}(\tau^{o}_j)$. This set defines a marked Riemann surface (S^{μ},α^{μ}) with the representative point $x = \underline{\Phi}(\mu)$. Thus (cf. §13, E)) w^{μ} induces a homeomorphism $f^{\mu}: S \longrightarrow S'$ which maps (S^{o},α^{o}) onto (S^{μ},α^{μ}). Since w^{μ} is orientation preserving, so is f^{μ}. Hence $\underline{\Phi}(\mu) \in C$ and we have defined a mapping $\underline{\Phi}: M \longrightarrow C$. We note that

$$(14.2) \qquad K[f^{\mu}] = \frac{1 + \|\mu\|}{1 - \|\mu\|} .$$

C) Lemma 1. <u>The mapping $\underline{\Phi}: M \longrightarrow C$ is sequentially continuous and onto.</u>

Proof. Assume that $\mu_n \to \mu$ in M. By §4, E) and, in particular, by the remark made at the end of that paragraph, we conclude that $w^{\mu_n} \longrightarrow w^{\mu}$ in the topology of W^*

(cf. §5, A)). This implies that the fixed points and invariants of A^{μ_n} converge to those of A^{μ} and proves the continuity of our mapping.

Now let x be a given point of C, $(A_1,\ldots,A_{2g})$ the corresponding normalized set of non-Euclidean translations and (S,α) the marked Riemann surface defined by it. By the definition of C the marked Riemann surfaces (S^o,α^o) and (S,α) are similarly oriented. Hence (cf. §9, D)) there exists a quasiconformal mapping $f : (S^o,\alpha^o) \rightarrow (S,\alpha)$. By §7, C) and D) the mapping f is induced by a homeomorphism $w : U \rightarrow U$. This homeomorphism is quasiconformal and must satisfy the relations $A_j = wA_j^o w^{-1}, j = 1,2,\ldots,2g$. In particular, it must satisfy the normalization condition (14.1). Hence w is of the form $w^{\mu}, \mu \in M$, and $x = \Phi(\mu)$.

D) <u>The regular quadratic differentials on a closed Riemann surface S of genus $g > 1$ form a real linear vector space of dimension $6g-6$.</u>

This is a classical corollary of the Riemann-Roch theorem. Indeed, let $\omega(z)dz$ be some holomorphic differential of type (1,0) on S (Abelian differential of the first kind). Every regular quadratic differential on S is of the form $H(\omega dz)^2$ where H is a meromorphic function and where the divisor of $H\omega^2$ is integral (i.e., $H\omega^2$ has no poles). The number of linearly independent functions satisfying this condition can be computed by the Riemann-Roch theorem and equals 6g-6.

E) It follows from D) that there exist $6g-6$ linearly independent functions, $\phi_1(z),\ldots,\phi_{6g-6}(z), z \in U$, which are holomorphic solution of the functional equation

$$(14.3) \qquad \phi(A_j^o(z)) = \frac{\phi(z)}{(dA_j^o(z)/dz)^2}, \quad j = 1,2,\ldots,2g$$

(cf. §8, B)). Using these functions we define a mapping of the open unit ball B in E_{6g-6} into the space M by setting $\mu = \underline{\Psi}(\xi)$ where $\xi = (\xi_1,\ldots,\xi_{6g-6})$ is a point in B and $\mu(z)$ the function $\mu(z) \equiv 0$ if $\xi = 0$ and the function

$$(14.4) \quad \mu(z) = \Big(\sum_{j=1}^{6g-6} \xi_j^2\Big)^{1/2} \Big(\sum_{j=1}^{6g-6} \xi_j\phi_j(z) \Big/ \Big|\sum_{j=1}^{6g-6} \xi_j\phi_j(z)\Big|\Big)$$

if $\xi \neq 0$ (cf. §8, B). We note that $\|\underline{\Psi}(\xi)\| = |\xi|$.

Lemma 2. <u>The mapping $\underline{\Psi} : B \to M$ is sequentially continuous.</u>

The proof is trivial.

F) Set $\Omega = \underline{\Phi}\,\underline{\Psi}$. The mapping $\Omega : B \to C$ has the following significance. If $x = \Omega(\xi)$, then there exists a Teichmüller mapping f_o of (S^o, a^o) onto the marked Riemann surface represented by the point x. For this mapping we have (cf. §7, B))

$$(14.5) \qquad K[f_o] = \frac{1+|\xi|}{1-|\xi|}.$$

Noting Lemmas 1 and 2, we obtain

Lemma 3. The mapping Ω: $B \longrightarrow C$ is continuous.

Lemma 4. The mapping Ω: $B \longrightarrow C$ is one-to-one.

Proof. If $x = \Omega(\xi_1) = \Omega(\xi_2)$ and $\xi_1 \neq \xi_2$, then there exist two distinct Teichmüller mappings of (S^o, α^o) onto the marked Riemann surface represented by x. This is impossible by the corollary to Theorem I (cf. §10, A)).

Lemma 5. The mapping Ω: $B \longrightarrow C$ is a homeomorphism and $\Omega(B)$ is open.

This follows from Lemmas 3 and 4 and Brouwer's theorem on invariance of domain.

G) Theorem II is contained in the following

Lemma 6. The mapping Ω: $B \longrightarrow C$ is onto, so that C is homeomorphic to E_{6g-6}.

In fact, this lemma implies that for any marked Riemann surface (S, α) of genus g which is similarly oriented to (S^o, α^o) there exists a Teichmuller mapping f_o: $(S^o, \alpha^o) \longrightarrow (S, \alpha)$.

Before proving this lemma we note

Lemma 7. If $\Omega(\xi) = \Phi(\mu)$, then $\|\mu\| \geq |\xi|$

Proof. Set $x = \Omega(\xi)$ and let (S, α) be the marked Riemann surface represented by this point. Then there exists a quasiconformal mapping f^{μ}: $(S^o, \alpha^o) \longrightarrow (S, \alpha)$ satisfying (14.2) and a Teichmüller mapping f_o: $(S^o, \alpha^o) \longrightarrow (S, \alpha)$ satisfying (14.5). The assertion follows from Theorem I.

H) Now we prove Lemma 6. Let x_1 be a given point in C. By Lemma 1 there is a $\mu \varepsilon M$ with $\Phi(\mu) = x_1$. For every $t, 0 \leq t \leq 1$, we have $t\mu \varepsilon M$. Set $x_t = \Phi(t\mu)$. The point x_t depends continuously on t, by virtue of Lemma 3.

Let Θ denote the set of those values of t for which $x_t \varepsilon \Omega(B)$. This set is not empty since it contains the origin: x_0 is the representative point of (S^0, α^0). The set Θ is open in view of Lemma 5. We want to show that it is closed, since this would imply that it contains 1, so that $x_1 \varepsilon \Omega(B)$.

Assume that there is a sequence $\{t_j\}$ in Θ with $t_j \rightarrow t_\infty$. Then there exists a sequence of points $\{\xi_j\}$ in B with $x_{t_j} = \Omega(\xi_j)$. By Lemma 7 $|\xi_j| \leq t_j \|\mu\| \leq \|\mu\| < 1$. Hence we may assume, selecting if need be a subsequence, that $\xi_j \rightarrow \xi_\infty \varepsilon B$. By Lemma 3 we have that $\Omega(\xi_\infty) = \lim \Omega(\xi_j) = \lim x_{t_j} = x_{t_\infty}$. Hence $t_\infty \varepsilon \Theta$, q.e.d. [3]

I) The following remarks are a digression from our main subject.

With the notations of A) and B) set $\hat{w}^\mu(z) = \overline{-w^\mu(z)}$. Then $\hat{w}^\mu : U \rightarrow U$ is an orientation-reversing homeomorphism and one sees easily that it induces a mapping $\hat{f}^\mu$ of (S^0, α^0)

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3 The reader will notice that we do not need the fact that the dimension δ of the space of regular quadratic differentials on S is 6g-6 but only the inequality $\delta \geq 6g-6$. On the other hand, Ahlfors needs for his variational proof, the fact that $\delta < \infty$.

onto a marked Riemann surface $(\hat{S}^\mu, \hat{\alpha}^\mu)$ with an opposite orientation. Let $\sigma_j^{\ \mu}, \tau_j^{\ \mu}, \lambda_j^{\ \mu}$ and $\hat{\sigma}_j^{\ \mu}, \hat{\tau}_j^{\ \mu}, \hat{\lambda}_j^{\ \mu}$ be the fixed points and invariants of the normalized sets $(A_1^\mu, \ldots, A_{2g}^\mu)$ and $(\hat{A}_1^\mu, \ldots, \hat{A}_{2g}^\mu)$ defining the marked Riemann surfaces (S^μ, α^μ) and $(\hat{S}^\mu, \hat{\alpha}^\mu)$, respectively. Since w^μ is orientation preserving and satisfies (14.1), $\sigma_{2g-1}^\mu \sigma_{2g-1}^o > 0$. On the other hand, we have that $\hat{\sigma}_j^{\ \mu} = -\sigma_j^{\ \mu}, \hat{\tau}_j^{\ \mu} = -\tau_j^{\ \mu}, \hat{\lambda}_j^{\ \mu} = \lambda_j^{\ \mu}$. Thus we obtain the following conclusions.

(1) Let $(A_1, \ldots, A_{2g})$ and $(B_1, \ldots, B_{2g})$ be normalized sets of non-Euclidean translations. The marked Riemann surfaces defined by these sets are similarly oriented if and only if $\sigma(A_{2g-1})\, \sigma(B_{2g-1}) > 0$

(2) The set C^* is obtained from the set C by changing the sign of $4g-4$ coordinates in E_{6g-6}.

15. The space of marked Riemann surfaces

The preceding considerations contain, implicitely, the proof of Theorem III. For the sake of completeness, we spell out the details.

A) Let C have the same meaning as in the preceding section. The points of C are thus representative points of marked Riemann surfaces of fixed genus $g > 1$, all with the same orientation. The Teichmüller distance, $d(x_1, x_2)$, between two points of C is, by definition, the Teichmüller

distance between the corresponding marked Riemann surfaces (cf. §10, C)). By Lemmas 1, 5, and 6 of the preceding section, we know that the mapping Ω establishes a homeomorphism between the unit ball B in E_{6g-6} and the set $C \subset E_{6g-6}$, with respect to the Euclidean distance function $|x_1 - x_2|$. Hence Theorem III will be proved once we establish the topological equivalence of the Euclidean and Teichmüller metrics in C.

B) Let $x_n, n = 0,1,2,\ldots,$ be points on C. Without loss of generality we may assume that x_o is the representative point of the marked Riemann surface (S^o, α^o) considered in §14. We set $\xi_n = \Omega^{-1}(x_n)$ and note that $\xi_o = 0$.

We must show that each of the two relations: $d(x_o,x_n) \to 0$. $|x_o - x_n| \to 0$, implies the other. Set, using the notations of the preceding section, $\mu_n = \Psi(\xi_n)$. Then the Teichmüller mapping $f_{o,n}$ of (S^o,α^o) onto the marked Riemann surface represented by x_n has as its Beltrami differential μ_n. Hence (cf. §14, E)) $K[f_{o,n}] = (1+|\xi_n|)/(1-|\xi_n|)$ so that (cf. §10, C)) $d(x_o,x_n) = \log[(1+|\xi_n|)/(1-|\xi_n|)]$. Thus $d(x_o,x_n) \to 0$ if and only if $\xi_n \to 0$ and since Ω is a homeomorphism, if and only if $|x_o - x_n| \to 0$.

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